Study Guide for Ph.D. Examination in Geometry and Topology (Math 5310)

Point-set topology

Definitions: Topological space, metric space.
Examples: Intervals in $\mathbb{R}$ (open, closed, half-open), $\mathbb{R}^\times$, $\mathbb{C}^\times$, subsets of $\mathbb{R}^n$, S^n, D^n, $\mathbb{P}^n(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{R})$, discrete topology, trivial topology, finite-complement topology.
Related concepts: Interior, closure, boundary, limit of a sequence, basis of a topology, fineness of a topology, second countable spaces.
Maps: Continuous maps, homeomorphisms, examples of homeomorphisms. Open maps, closed maps.
Induced topologies: Subspace topology, quotient topology (and its universal property), product topology, disjoint unions. Many examples for quotient topology.
Separation Axioms: Hausdorff, normal, Urysohn's Lemma.
Compactness: Definition, statement of Heine-Borel (without proof), simple properties. Applications: Hausdorff and compact $\Rightarrow$ normal; maximum/minimum for real-valued functions; any map from compact to Hausdorff induces a homeomorphism (and variations on this statement). Tychonoff's theorem for finite products. Tychonoff's theorem for infinite products (without proof). Sequentially compact. For second countable spaces, sequentially compact $\Leftrightarrow$ compact.
Connectedness: Several equivalent definitions of connectedness. Z connected $\Rightarrow \overline{Z}$ is connected. X is connected and $f\colon X \to Y$ continuous $\Rightarrow f(X)$ connected. X, Y connected $\Rightarrow X \times Y$ connected. Connected components. Path-connectedness. Locally path-connected.

Surfaces

Definitions: Topological manifolds, surfaces.
Examples: Sphere, projective plane, torus, Klein bottle.
Constructions: Connected sum. Polygon representations of surfaces.
Classification of Surfaces: Classification of polygon representations. Euler characteristic of polygon representations.

Fundamental Group

Basics: Homotopy of paths. Construction of the fundamental group. Simply-connected spaces.
Examples: $\pi_1(S^1)$. Brouwer's fixed point theorem and similar applications. π_1 of a product.
Induced Maps: Examples. Fundamental groups of spheres. Deformation retracts.
Seifert-van Kampen: Free products of groups, statement of Seifert-van Kampen (without proof).
Covering Spaces: Definition. Lifting of paths and of homotopies. Lifting criterion in terms of the fundamental group.
Classification of coverings: Universal covering, classification of connected coverings via subgroups of π_1 (with or without choice of basepoint). Classification of coverings via permutation actions of π_1.
Galois coverings: Deck transformations, normal coverings. Group action and coverings.

References:
T. W. Gamelin and R. E. Greene, *Introduction to Topology*, 2nd ed., Dover, 1999. Section 1.1, Chapter 2, Sections 3.1–3.7.
A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, 2002. Chapter 1. Skip "Applications to Cell Complexes" in 1.2. Also available online
J. Lee, *Introduction to Topological Manifolds*, Springer-Verlag, 2000. Chapters 2–4, 6–12. Skip 2nd half of chapter 7.
J. Munkres, *Topology*, 2nd ed., Prentice Hall, 2000. Chapters 2–5, 9, 11–14. Skip sections 34–36, 38, 75, and probably more.