Study Guide for Ph.D. Examination in Algebra (Math 5210)

Questions on the algebra prelim exam will be drawn from the topics below. **Note**: Math 5210 will not necessarily cover every topic listed, so some amount of self-study may be required, depending on your background.

Group Theory: cyclic groups, homomorphisms, subgroup, quotient group, direct product. Isomorphisms and automorphisms. Group actions (esp. conjugation and left multiplication), orbit-stabilizer formula, applications to p-groups and theorems of Cauchy and Sylow. Semidirect products. Cyclic decomposition and classification of finite abelian group (statement only).

Examples: $\mathbf{Z}/m\mathbf{Z}$, $(\mathbf{Z}/m\mathbf{Z})^\times$, μ_m, D_n, A_n, S_n, $\mathrm{GL}_n(\mathbf{R})$, $\mathrm{SL}_n(\mathbf{R})$, $\mathrm{Aff}(\mathbf{R})$, and similar finite matrix groups over $\mathbf{Z}/m\mathbf{Z}$ (especially $\mathbf{Z}/p\mathbf{Z}$).

Ring Theory: Characteristic, integral domain, field. Homomorphisms, subring, ideals and quotient rings, principal ideals, prime ideals and maximal ideals, nilpotent elements, Zorn's lemma. Chinese remainder theorem for rings. Fraction field, Euclidean domain, PID, UFD, irreducibility tests for polynomials (reduction mod p, Eisenstein criterion in $\mathbf{Z}[X]$ and $R[X]$ when R is a UFD).

Examples: $\mathbf{Z}$, $\mathbf{Z}/m\mathbf{Z}$, $F[X]$, $F[X]/(f)$, $\mathbf{Z}[i]$ and other $\mathbf{Z}[\sqrt{d}]$, $F(X)$, $\mathbf{Q}(\sqrt{d})$, $\mathbf{Z}[X]$, $F[X,Y]$, quotient rings $\mathbf{Z}[\sqrt{d}]/I$, $\mathbf{Z}[X]/I$, $F[X,Y]/I$.

Linear Algebra: Vector spaces: subspace and quotient space, basis, dimension, linear transformations (including matrix representation using a basis). Dual spaces, dual of linear map, double duality. Determinant and trace, characteristic polynomial, eigenvalues and eigenvectors. Inner product spaces, orthogonal and orthonormal bases, adjoint of linear map, spectral theorem for self-adjoint operator (over $\mathbf{R}$).

Examples of vector spaces: F^n, $F[X]$, polynomials in $F[X]$ of degree at most d, $\mathrm{M}_n(F)$, $\mathrm{Hom}_F(V,W)$, $\mathbf{C}$ as real vector space.

Examples of inner product spaces: $\mathbf{R}^n$ with $\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}\cdot\mathbf{w}$, $\mathrm{M}_n(\mathbf{R})$ with $\langle A,B\rangle = \mathrm{Tr}(AB^\top)$, $C[0,1]$ with $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$.

References:

D. Dummit and R. Foote, *Abstract Algebra*, 3rd ed., Wiley, 2003.

K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, 2nd ed., 1971.

T. Hungerford, *Algebra*, Springer-Verlag, 1980.

S. Lang, *Algebra*, 3rd revised ed., Springer-Verlag, 2005.

J. Rotman, *Advanced Modern Algebra*, Prentice Hall, 2002.

Remark. This study guide is effective starting with the January 2021 algebra prelim exam.